THE THEORY OF QUALITY OF NONLINEAR CONTROL SYSTEMS

(K TEORII KACHESTVA NELINEINYKH REGULIRUEMYKH SISTEM)

PMM Vol.23, No.5, 1959, pp. 971-974

CHZHAN SY-IN
(Moscow-Shen'ian)

(Received 12 February 1959)

The system considered here belongs to a large class of control systems and its motion is described by the differential equations of the form [1]

$$\dot{\eta}_k = \sum_{\alpha=1}^{m} b_{k\alpha}\eta_\alpha + n_k\mu \qquad (k = 1, \ldots, m)$$

$$v^2\ddot{\mu} + w\dot{\mu} + s\mu = f^*(\sigma), \qquad \sigma = \sum_{\alpha=1}^{m} p_\alpha\eta_\alpha - r\mu \tag{1}$$

Here η_k are generalized coordinates of the controlled object, $b_{k\alpha}$ are constants of the controlled object, μ is a coordinate of the controller, n_k are constant parameters of the controller, v, w, s, generally speaking, are known functions of the variables μ, $\dot{\mu}$, σ, in special cases s, v may be constants or zero, σ may be a combined (summed) controlling pulse signal, p_α, r are constants of the controller, $f^*(\sigma)$ is nonlinear characteristic of the servomotor.

Let

$$\frac{1}{w} f^*(\sigma) = f(\sigma) \tag{2}$$

In most control systems $f(\sigma)$ belongs to one of the two classes of the following functions

$$f(\sigma) = 0 \quad \text{for} \quad |\sigma| \leqslant \sigma_*, \qquad \sigma f(\sigma) > 0 \quad \text{for} \quad |\sigma| > \sigma_* \tag{3}$$

where σ_* is some fixed non-negative number characterizing the dead zone of the controller. Sometimes $f(\sigma)$ satisfies the following conditions:

$$\sigma_* = 0, \quad \left[\frac{df(\sigma)}{d\sigma}\right]_{\sigma=0} \geqslant h > 0, \quad \varphi(\sigma) = f(\sigma) - h(\sigma), \qquad \sigma\varphi(\sigma) > 0 \text{ for } \sigma \neq 0 \tag{4}$$

where h is a given constant. In a special case one has

$$f(\sigma) = +Q \text{ for } \sigma > 0, \quad f(\sigma) = 0 \text{ for } \sigma = 0, \quad f(\sigma) = -Q \text{ for } \sigma < 0 \quad (5)$$

For the sake of simplicity let us restrict ourselves to the case when $\nu^2 = 0$. In addition, let us use the notation

$$\rho_{m+1} = \frac{s}{w} \quad (6)$$

The system (1) shall assume the form

$$\dot{\eta}_k = \sum_{\alpha=1}^{m} b_{k\alpha}\eta_\alpha + n_k\mu \ (k = 1, \ldots, m), \quad \dot{\mu} = -\rho_{m+1}\mu + f(\sigma), \quad \sigma = \sum_{\alpha=1}^{m} p_\alpha\eta_\alpha - r\mu \quad (7)$$

Eliminating μ by means of the equation $\sigma = \Sigma p_\alpha \eta_\alpha - r\mu$ $(r \neq 0)$ and using the following notation

$$b_{k\alpha}^{\circ} = b_{k\alpha} + \frac{n_k p_\alpha}{r} \qquad (\alpha, k = 1, \ldots, m)$$

$$\sum_{\alpha=1}^{m} p_\alpha b_{\alpha\beta}^{\circ} + \rho_{m+1} p_\beta = p_\beta^{\circ}, \qquad \sum_{\alpha=1}^{m} \frac{p_\alpha \eta_\alpha}{r} + \rho_{m+1} = \rho^{\circ} \quad (8)$$

the system (7) may be reduced to the form

$$\dot{\eta}_k = \sum_{\alpha=1}^{m} b_{k\alpha}^{\circ}\eta_\alpha - \frac{n_k}{r}\sigma \quad (k = 1, \ldots, m), \qquad \dot{\sigma} = \sum_{\alpha=1}^{m} p_\alpha^{\circ}\eta_\alpha - \rho^{\circ}\sigma - rf(\sigma) \quad (9)$$

Let us introduce linear non-singular transformation

$$\chi_s = \sum_{\alpha=1}^{m} C_\alpha^{(s)}\eta_\alpha \quad (s = 1, \ldots, n) \quad (10)$$

and select coefficients $C_\alpha^{(s)}$ such that

$$-r_s C_\beta^{(s)} = \sum_{\alpha=1}^{m} C_\alpha^{(s)} b_{\alpha\beta}^{\circ} \quad (\beta, s = 1, \ldots, m), \qquad -r = \sum_{\alpha=1}^{m} C_\alpha^{(s)} n_\alpha \quad (11)$$

where r_s are roots of the following equation

$$D^{\circ}(r) = \left\| \begin{matrix} b_{11}^{\circ} + r & b_{21}^{\circ} \ldots b_{m1}^{\circ} \\ b_{1m}^{\circ} & b_{2m}^{\circ} \ldots b_{mm}^{\circ} + r \end{matrix} \right\| = 0 \quad (12)$$

Then we reduce the system (9) to the canonic form

$$\dot{\chi}_k = -r_k\chi_k + \sigma \quad (k = 1, \ldots, m), \qquad \dot{\sigma} = \sum_{k=1}^{m} \beta_k^{\circ}\chi_k - \rho^{\circ}\sigma - rf(\sigma) \quad (13)$$

Here

$$\beta_k^{\circ} = \sum_{\alpha=1}^{m} D_k^{\circ(\alpha)} p_\alpha^{\circ} \qquad \left(\eta_k = \sum_{\alpha=1}^{m} D_k^{\circ(k)}\chi_\alpha\right) \qquad (k = 1, \ldots, m) \quad (14)$$

According to (8) and (12), the parameters of the controller n_k, p_α, r may be selected such that for every r the following will be valid

$$\operatorname{Re} r_s > 0 \qquad (s = 1, \ldots, m) \quad (15)$$

Let us consider a case when $D^0(r) = 0$ has only simple roots. If all the roots are real, then the problem of quality will be solved by means of an equation of the form (13). If $D(r) = 0$ has s real roots $r_i (i = 1, \ldots, s)$ and $2k$ complex roots $\lambda_i \pm i\mu_j (j = s + 1, \ldots, m - k)$ then instead of (13) the equations of the following form will be used

$$\dot{\chi}_i = -r_i\chi_i + \sigma \quad (i = 1, \ldots, s), \qquad \dot{\chi}_j = -\lambda_j\chi_j + \mu_j\chi_{j+k} + 2\sigma$$
$$\dot{\chi}_{j+k} = -\lambda_j\chi_{j+k} - \mu_j\chi_j \qquad (j = s + 1, \ldots, m - k) \tag{16}$$
$$\dot{\sigma} = \sum_{i=1}^{s}\beta_i{}^{\circ}\chi_i + \sum_{j=s+1}^{m-k}(\beta_j{}^{\circ\circ}\chi_j + \beta_{j+k}{}^{\circ\circ}\chi_{j+k}) - \rho^{\circ}\sigma - rf(\sigma)$$

Here $\beta_j{}^{00}$, $\beta_{j+k}{}^{00}$ are real numbers.

In order to investigate transient response let us consider a sphere in the phase space of the variables χ_k $(k = 1, \ldots, m)$, σ

$$R^2 = \chi_1{}^2 + \ldots + \chi_m{}^2 + \sigma^2 \tag{17}$$

the radius of which at $t_0 = 0$ is equal to $R(0)$. Here all the parameters of the system are given and it is desired to find the time t^* required for the radius $R(t)$ to decrease e^a times, where a is a given positive number, i.e.

$$\frac{R^2(t^*)}{R^2(0)} = e^{-2a} \tag{18}$$

The inverse problem would be to determine conditions for the system parameters such that t^* would not exceed some specified t^*. Let us consider a function [2]

$$V = e^{\alpha t}(\chi_1{}^2 + \ldots + \chi_m{}^2 + \sigma^2) \tag{19}$$

Here α is a constant and is left undefined.

Let all the roots of $D^0(r)$ be simple and real. By virtue of (13) we have

$$\frac{dV}{dt} = \alpha e^{\alpha t}(\chi_1{}^2 + \ldots + \chi_m{}^2 + \sigma^2) + e^{\alpha t}\left[2\chi_1\frac{d\chi_1}{dt} + \ldots + 2\chi_m\frac{d\chi_m}{dt} + 2\sigma\frac{d\sigma}{dt}\right] =$$
$$= e^{\alpha t}\left[(\alpha - 2r_1)\chi_1{}^2 + \ldots + (\alpha - 2r_m)\chi_m{}^2 + (\alpha - 2\rho^{\circ})\sigma^2 + 2\sigma\sum_{k=1}^{m}(1 + \beta_k{}^{\circ})\chi_k\right] +$$
$$+ e^{\alpha t}[-2r\sigma f(\sigma)] \tag{20}$$

Recalling that the function $f(\sigma)$ may have a form (3) or (4). Assuming (4) we obtain

$$\frac{dV}{dt} = e^{\alpha t}\Big[(\alpha - 2r_1)\chi_1{}^2 + \ldots + (\alpha - 2r_m)\chi_m{}^2 + (\alpha - 2\rho^{\circ} - 2hr)\sigma^2 +$$
$$+ 2\sigma\sum_{k=1}^{m}(1 + \beta_k)\chi_k\Big] + e^{\alpha t}[-2r\sigma\varphi(\sigma)] \tag{21}$$

Let us choose α such that

$$dV/dt \leqslant 0 \tag{22}$$

Let us note that $\sigma\phi(\sigma) > 0$ and that the number r, in general, is positive. Therefore, if the quadratic form

$$H = (\alpha - 2r_1)\chi_1^2 + \ldots + (\alpha - 2r_m)\chi_m^2 + (\alpha - 2\rho^\circ - 2hr)\sigma^2 + 2\sigma\sum_{k=1}^{m}(1+\beta_k)\chi_k \tag{23}$$

is non-positive, we shall have the condition (22); the conditions of the non-negativness of the form H are as follows:

$$\Delta(\alpha) = \begin{vmatrix} 2r_1 - \alpha & 0 & \ldots & 0 & -(1+\beta_1^\circ) \\ 0 & 2r_2 - \alpha & \ldots & 0 & -(1+\beta_2^\circ) \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ 0 & 0 & & 2r_m - \alpha & -(1+\beta_m^\circ) \\ -(1+\beta_1^\circ) & -(1+\beta_2^\circ) & \ldots & -(1+\beta_m^\circ) & 2(\rho^\circ + hr) - \alpha \end{vmatrix} \tag{24}$$

all minors of this determinant must be non-negative, i.e.

$$A\begin{pmatrix} i_1 i_2 \ldots i_p \\ i_1 i_2 \ldots i_p \end{pmatrix} \geqslant 0, \qquad \begin{pmatrix} 1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant m \\ p = 1, \ldots, m) \end{pmatrix} \tag{25}$$

Let us denote through α^* such value of α for which the conditions (25) are satisfied. Then from (22) we shall have

$$e^{\alpha^* t}(\chi_1^2 + \ldots + \chi_m^2 + \sigma^2) \leqslant e^{\alpha^* t_0}(\chi_{10}^2 + \ldots + \chi_{m0}^2 + \sigma_0^2) \tag{26}$$

From (26) and (18) we can find t^* (assuming $t_0 = 0$):

$$e^{\alpha^* t^*} \leqslant e^{2a}, \quad \text{or} \quad t^* \leqslant \frac{2a}{\alpha^*} \tag{27}$$

If instead of (25) we assume the following more rigid conditions

$$\Delta_1 > 0, \qquad \Delta_2 > 0, \ldots, \qquad \Delta_m = \Delta > 0 \tag{28}$$

then $-H$ will be positive. In this case $dV/dt < 0$, and consequently

$$t^* < \frac{2a}{\alpha^{**}} \tag{29}$$

Here α^{**} is a number ensuring fulfilment of (28). This number can be chosen as follows. Without losing generality let us assume that

$$r_1 < r_2 < \ldots < r_m < \rho^\circ + hr \tag{30}$$

(of course, $\rho^0 + hr$ may take on any intermediate values among the numbers r_1, r_2, ...). One can see that according to Sylvester all the roots of equation $\Delta(\alpha) = 0$ (24) are real. Furthermore, Letov [1] proved that the smallest root of equations $\Delta(\alpha) = 0$ is less or equal to $2r_1$, i.e. $\alpha_{\min} \leqslant 2r_1$. From (24) and (28) we can see that this value will reach its limit for the conditions (28). Therefore, by virtue of (29), we shall have

$$t^* < \frac{2a}{2r_1} = \frac{a}{r_1} \tag{31}$$

In case when $D(r) = 0$ has complex roots, the analysis is carried out in an analogous fashion.

Example. Let us consider the problem of Bulgakov [3]. The equations of motion are as follows

$$T^2\ddot{\psi} + U\dot{\psi} + K\psi + \mu = 0, \quad \dot{\mu} = f^*(\sigma), \qquad \sigma = a\psi + E\dot{\psi} + G\ddot{\psi} - \frac{1}{l}\mu \tag{32}$$

Let us introduce the notation

$$\psi = \eta_1, \quad \dot{\psi} = \sqrt{r}\,\eta_2, \quad \mu = i\xi, \quad t = \frac{\tau}{\sqrt{r}}, \quad p = \frac{U}{T^2}, \quad q = \frac{K}{T^2},$$

$$r = \frac{i}{T^2}, \quad n_2 = -1, \quad i = \frac{lT^2}{T^2 + lG^2}, \quad f(\sigma) = \frac{1}{i\sqrt{r}} f^*(\sigma), \quad b_{21} = -\frac{q}{r} \tag{33}$$

$$b_{22} = -\frac{p}{\sqrt{r}}, \quad p_1 = a - qG^2, \quad p_2 = (E - pG^2)\sqrt{r}, \quad p_3 = -1$$

Then (32) will assume the form

$$\dot{\eta}_1 = \eta_2, \quad \dot{\eta}_2 = b_{21}\eta_1 + b_{22}\eta_2 + n_2\xi, \quad \dot{\xi} = f(\sigma), \quad \sigma = p_1 n_1 + p_2\eta_2 - \xi \tag{34}$$

A dot here denotes a derivative with respect to dimensionless time τ. Eliminating ξ, we obtain

$$\dot{\eta}_1 = \eta_2, \qquad \dot{\eta}_2 = b_{21}{}^\circ\eta_1 + b_{22}{}^\circ\eta_2 + \sigma,$$
$$\dot{\sigma} = p_1{}^\circ\eta_1 + p_2{}^\circ\eta_2 - \rho^\circ\sigma - f(\sigma)$$

where

$$b_{21}{}^\circ = b_{21} - p_1, \quad b_{22}{}^\circ = b_{22} - p_2, \quad p_1{}^\circ = b_{21}{}^\circ p_2, \quad p_2{}^\circ = p_1 + b_{22}{}^\circ p_2, \quad p^\circ = -p_2$$

In the case considered we have

$$D(r) = \begin{vmatrix} r & b_{21}{}^\circ \\ 1 & r + b^\circ{}_{22} \end{vmatrix} = 0$$

the roots of this equation will be

$$r_{1,2} = \frac{1}{2}\frac{U + lE}{\sqrt{l(T^2 + lG^2)}} \pm \sqrt{\frac{1}{4}\frac{(U + lE)^2}{l(T^2 + lG^2)} - \frac{K + al}{l}}$$

Here

$$\frac{1}{4}\,\frac{(U + lE)^2}{l(T^2 + lG^2)} < \frac{K + al}{l}$$

therefore the roots r_1, r_2 are conjugate complex. Let us write

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu.$$

By means of a linear transformation, equation (34) may be reduced to a canonic form

$$\dot{\chi}_1 = -\lambda\chi_1 + \mu\chi_2 + 2\sigma, \qquad \dot{\chi}_2 = -\lambda\chi_2 - \mu\chi_1$$
$$\dot{\sigma} = p^\circ{}_2\chi_1 - \frac{1}{\mu}(p_1{}^\circ - \lambda p_2{}^\circ)\chi_2 - \rho^\circ\sigma - f(\sigma)$$

Let us consider the function $V = e^{at}(\chi_1^2 + \chi_2^2 + \sigma^2)$. In this case we have

$$\Delta(\alpha) = \begin{vmatrix} 2\lambda - \alpha & 0 & -(2+\beta_1^*) \\ 0 & 2\lambda - \alpha & -\beta_2^* \\ -(2+\beta_1^*) & -\beta_2^* & 2(\rho^\circ + h) - \alpha \end{vmatrix}$$

where

$$p_2^\circ = \beta_1^*, \qquad -\frac{1}{\mu}(p_1^\circ - \lambda p_2^\circ) = \beta_2^*$$

If $\lambda < (\rho^0 + h)$, then the smallest root of $\Delta(\alpha) = 0$ will be $a_{\min} \leqslant 2\lambda$.

Let us determine time in accordance with (31):

$$\tau^* < \frac{a}{\lambda} = \frac{2a\sqrt{l(T^2 + lG^2)}}{U + lE}$$

But from (33) $\tau = t\sqrt{r}$ so that

$$t^* < \frac{2a\sqrt{l(T^2 + lG^2)}}{(U + lE)\sqrt{r}} = \frac{2a(T^2 + lG^2)}{U + l\varepsilon}$$

This result is the same as the one obtained by Letov for the same case.

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Translated by P.N.B.